JEE Advanced : Paper-2 (2015)

IMPORTANT INSTRUCTIONS

1. The question paper has three parts: **Physics**, **Chemistry** and **Mathematics**. Each part has three sections.
2. Section 1 contains 8 questions. The answer to each question is a single digit integer ranging from 0 to 9 (both inclusive).
   Marking Scheme: +4 for correct answer and 0 in all other cases.
3. Section 2 contains 8 multiple choice questions with one or more than one correct option.
   Marking Scheme: +4 for correct answer, 0 if not attempted and –2 in all other cases.
4. Section 3 contains 2 "paragraph" type questions. Each paragraph describes an experiment, a situation or a problem. Two multiple choice questions will be asked based on this paragraph. One or more than one option can be correct.

PART A: PHYSICS

SECTION 1

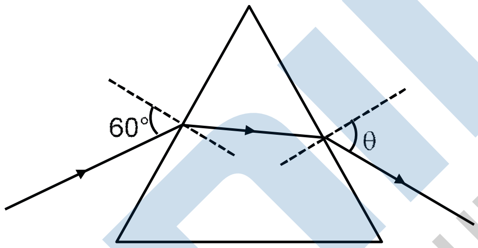
SECTION 1 (Maximum Marks: 32)

- This section contains EIGHT questions
- The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive
- For each question, darken the bubble corresponding to the correct integer in the ORS
- Marking scheme:

  +4 If the bubble corresponding to the answer is darkened

  0 In all other cases

1. A monochromatic beam of light is incident at 60° on one face of an equilateral prism of refractive index n and emerges from the opposite face making an angle θ(n) with the normal (see the figure). For $n = \sqrt{3}$ the value of θ is 60° and $\frac{d\theta}{dn} = m$. The value of m is



Ans. **[2]**

Sol. $\sin 60° = n \sin r_1$

$n \sin r_2 = \sin Q$

$n \sin(60° - r_1) = \sin Q$

$n(\sin 60° \cos r_1 - \cos 60° \sin r_1) = \sin Q$

$$\frac{\sqrt{3}n}{2}\cos r_1 - \frac{n}{2}\sin r_1 = \sin Q$$

$$\frac{\sqrt{3}n}{2}\sqrt{1-\frac{3}{4n^2}} - \frac{1}{2}\frac{\sqrt{3}}{2} = \sin Q$$

$$\frac{\sqrt{3}}{2}\frac{n}{2n}\sqrt{4n^2-3} - \sqrt{4n^2-3} - \frac{\sqrt{3}}{4} = \sin Q$$

$$\frac{4}{\sqrt{3}}\sin Q = \sqrt{4n^2-3} - 1$$

On differentiatig

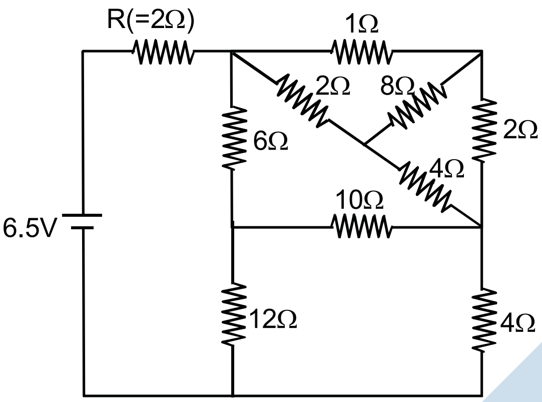
$$\frac{4}{\sqrt{3}}\cos Q\frac{dQ}{dn} = \frac{1}{2\sqrt{4n^2-3}}8n$$

Putting the values

$$\frac{4}{\ }\frac{1}{\ }\frac{d\theta}{\ } = \frac{1}{\ }8\sqrt{3} = \frac{4\sqrt{3}}{\ } = \frac{4}{\ }$$

$\frac{d\theta}{dn} = 2$ m = 2 Ans.

2. In the following circuit, the current through the resistor R (= 2Ω) is I Amperes. The value of I is

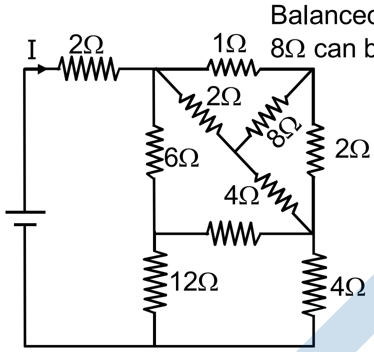


Ans. **[1]**

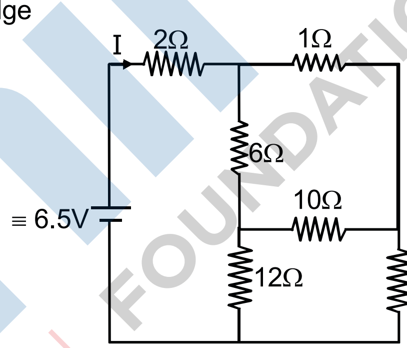
current electricity

Sol.

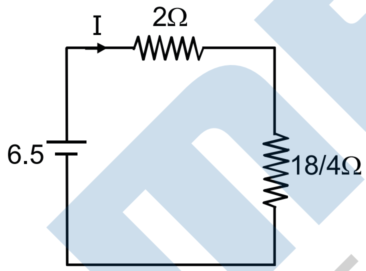
Balanced Wheatstone bridge
8Ω can be removed



$\equiv$ 6.5V



Balanced so WSB



$I = \frac{6.5}{2 + (18/4)} = \frac{6.5 \times 4}{26} = 1A$

3. An electron in an excited state of Li^{2+} ion has angular momentum $3h/2\pi$. The de Broglie wavelength of the electron in this state is $p\pi a_0$ (where a_0 is the Bohar radius). The value of p is

Sol. $L = \frac{nh}{2\pi} = \frac{3h}{2\pi} \Rightarrow n = 3$

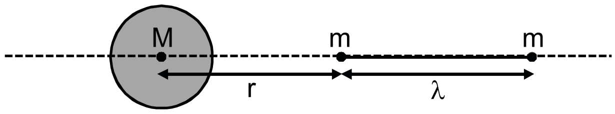
$mvr = \frac{nh}{2\pi}$

$P = \frac{nh}{2\pi r}$

$So, \lambda = \frac{h}{P} = \frac{2\pi r}{n} = \frac{2\pi}{3}\left(\frac{n^2}{2}a_0\right) \therefore = \frac{2\pi \times 3a_0}{3} = 2\pi a_0 = P\pi a_0$

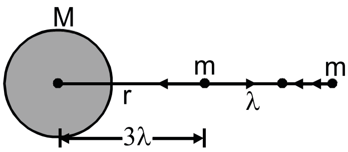
So, P = 2

4. A large spherical mass M is fixed at one position and two identical point masses m are kept on a line passing through the centre of M (see figure). The point masses are connected by a rigid massless rod of length ℓ and this assembly is free to move along the line connecting them. All three masses interact only through their mutual gravitational interaction. When the point mass nearer to M is at a distance r = 3ℓ from M, the tension in the rod is zero for $m = k\left(\frac{M}{288}\right)$. The value of k is:



Ans. **[7]**

Sol.



For tension in the rod to be zero.

Both masses should move with same acceleration, only due to gravitation alteraction

$$\frac{GMm}{9\ell^2} - \frac{Gm^2}{\ell^2} = \frac{GMm}{16\ell^2} + \frac{Gm^2}{\ell^2}$$

$$\frac{GMm}{\ell^2}\left\{\frac{7}{9\times16}\right\} = \frac{2Gm^2}{\ell^2}$$

$$m = \left(\frac{7M}{288}\right) = K\frac{M}{288} \text{ So} K = 7$$

5. The energy of a system as a function of time t is given as $E(t) = A^2\exp(-\alpha t)$, where $\alpha = 0.2\ s^{-1}$. The measurement of A has an error of 1.25%. If the error in the measurement of time is 1.50%, the percentage error in the value of E(t) at t = 5 s is

Ans. **[4]**

Sol. $E(t) = A^2e^{-\alpha t}$

$dE = 2Ae^{-\alpha t}\,dA + A^2(-a)e^{-\alpha t}.\,dt$

$$\frac{dE}{R} = \frac{2Ae^{-\alpha t}dA - \alpha A^2e^{-\alpha t}dt}{A^2e^{-\alpha t}}$$

$$\frac{2A}{A^2} = dA - \frac{\alpha A^2}{A^2}dt.$$

$$\frac{dE}{E} = \frac{2}{A}dA - \alpha dt$$

$$\frac{dE}{E} = \frac{2}{A}dA + \alpha dt = \text{(for errors)}$$

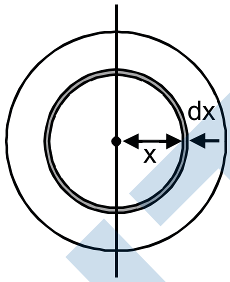
$$= 2\times\frac{1.25}{100} + 0.2\times5\times\frac{1.50}{100}$$

$$= \frac{2.5}{100} + \frac{1.50}{100} = \frac{4}{100} = 4\%$$

6. The densities of two solid spheres A and B of the same radii R vary with radial distance r as $\rho_A(r) = k\left(\frac{r}{R}\right)$ and $\rho_B(r) = k\left(\frac{r}{R}\right)^5$, respectively, where k is a constant. The moments of inertia of the individual spheres about axes passing through their centres are I_A and I_B, respectively. If $\frac{I_B}{I_A} = \frac{n}{10}$, the value of n is

Ans. **[6]**

Sol. $dm = \rho(x) \quad 4\pi x^2 dx = \frac{k}{R} x \; . \; 4\pi x^2 dx$



$$dI_A = \frac{2}{3}(dm)x^2 = \frac{2}{3}\frac{k}{R} 4\pi x^5 \, dx$$

$$dI_B = \frac{2}{3}\frac{K}{R^5} 4\pi x^9 \, dx$$

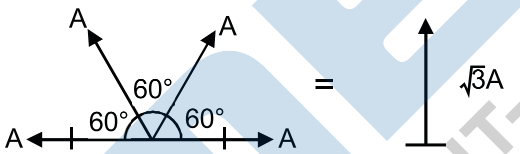
$$\frac{I_B}{I_A} = \frac{1}{R^4} \cdot \frac{R^{10}}{10} \cdot \frac{6}{R^6} = \frac{6}{10} = \frac{n}{10}$$

n = 6

7. Four harmonic waves of equal frequencies and equal intensities I_0 have phase angles 0, $\pi/3$, $2\pi/3$ and π. When they are superposed, the intensity of the resulting wave is nI_0. The value of n is

Ans. **[3]**

Sol.



$I_0 \propto A^2$

So, IR $\propto (\sqrt{3}A)^2 = 3I_0$

n = 3

8. For a radioactive material, its activity A and rate of change of its activity R are defined as $A = -\frac{dN}{dt}$ and $R = -\frac{dA}{dt}$, where N(t) is the number of nuclei at time t. Two radioactive sources P (mean life τ) and Q (mean life 2τ) have the same activity at t = 0. Their rates of change of activities at t = 2τ are R_P and R_Q, respectively. If $\frac{R_P}{R_Q} = \frac{n}{e}$, then the value of n is

Ans. **[2]**

Sol. $N = N_0 e^{-\lambda t}$

$$A = -\frac{dN}{dt} = \lambda N_0 e^{-\lambda t} = \lambda N$$

$$R = -\frac{dA}{dt} = +\lambda^2 N_0 e^{-\lambda t} = \lambda A = \lambda^2 N = \lambda^2 N_0 e^{-\lambda t}$$

Now, $\lambda_P N_{P0} = \lambda_Q N_0$

$$\frac{R_P}{R_Q} = \frac{\lambda_P}{\lambda_Q}\frac{A_P}{A_v} = 2$$

$$= 2\frac{e^{-2}}{e^{-1}} = \frac{2}{e}$$

n = 2

SECTION 2

(Maximum Marks : 32)

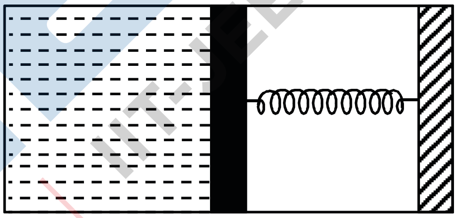
- This section contains EIGHT questions
- Each questions has FOUR options (A), (B), (C) and (D). ONE OR MORE THAN ONE of these four option(s) is (are) correct
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS
- Marking scheme:

  +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened

  0 If none of the bubbles is darkened

  –2 In all other cases

9. An ideal monoatomic gas is confined in a horizontal cylinder by a spring loaded piston (as shown in the figure). Initially the gas is at temperature T_1, pressure P_1 and volume V_1 and the spring is in its relaxed state. The gas is then heated very slowly to temperature T_2, pressure P_2 and volume V_2. During this process the piston moves out by a distance x. Ignoring the friction between the piston and the cylinder, the correct statement(s) is (are)



(A) If $V_2 = 2V_1$ and $T_2 = 3T_1$, then the energy stored in the spring is $\frac{1}{4}P_1V_1$

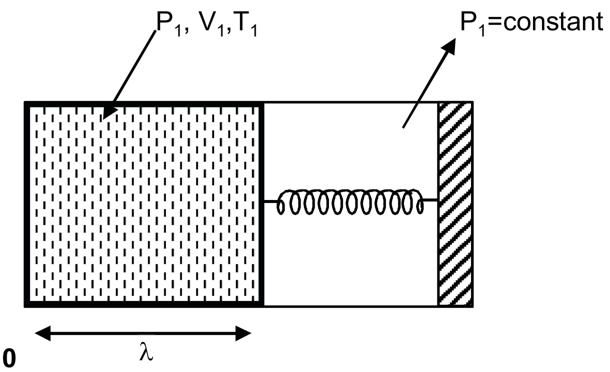
(B) If $V_2 = 2V_1$ and $T_2 = 3T_1$, then the change in internal energy is $3P_1V_1$

(C) If $V_2 = 3V_1$ and $T_2 = 4T_1$, then the work done by the gas is $\frac{7}{3}P_1V_1$

(D) If $V_2 = 3V_1$ and $T_2 = 4T_1$, then the heat supplied to the gas is $\frac{17}{6}P_1V_1$

Ans. [A, B, C]

Sol.



V_1 = Al, ℓ = initial length of segment containing monoatomic gas.

$P_2 = \frac{T_2}{V_2}\frac{P_1V_1}{T_1}$

(A) $P_2 = \frac{3T_1}{2V_1}\frac{P_1V_1}{T_1} = \frac{3}{2}P_1$

$P_1A + Kx = P_2A$

$kx = (P_2 - P_1)A = \left(\frac{3}{2} - 1\right)P_1A = \frac{P_1A}{2}$

as $v_2 = 2v_1$ so $x = \ell$

so $k\ell = \left(\frac{P_1A}{2}\right)$

energy of spring= $\frac{1}{2}kx^2 = \frac{1}{2}k\ell^2 = \frac{1}{2}\frac{P_1A\ell}{2} = \frac{P_1V_1}{4}$

increase in U, $\Delta U = \frac{3}{2}nR(2T_1) = 3nRT_1 = 3P_1V_1$

Hence $q = \Delta U + w = 3P_1V_1 + \frac{P_1v_1}{4} = \frac{13P_1v_1}{4}$

(C) For $v_2 = 3v_1$ and $T_2 = 4T_1$

$P_2 = \frac{4}{3}P_1 \quad \Rightarrow kx = \left(\frac{4}{3} - 1\right)P_1A = \frac{P_1A}{3}$

work done by gas = $P1(2\ell)A + \frac{1}{2}\frac{P_1A}{3}(2\ell) = 2P_1v_1 + \frac{P_1v_1}{3} = \frac{7}{3} = P_1V_1$

10. A fission reaction is given by ${}^{236}_{92}U \rightarrow {}^{140}_{54}Xe + {}^{94}_{38}Sr + x + y$, where x and y are two particles. Considering ${}^{236}_{92}U$ to be at rest, the kinetic energies of the products are denoted by K_{Xe}, K_{Sr}, K_x (2MeV) and K_y (2MeV), respectively. Let the binding energies per nucleon of ${}^{236}_{92}U$, ${}^{140}_{54}Xe$, and ${}^{94}_{38}Sr$ be 7.5 MeV, 8.5 MeV and 8.5 MeV, respectively. Considering different conservation laws, the correct option(s) is (are)

(A) x =n, y = n, K_{Sr} = 129 MeV, K_{Xe} = 86 MeV

(B) x = p, y = e^-, K_{Sr} = 129 MeV, K_{Xe} = 86 MeV

(C) x = p, y = n, K_{Sr} = 129 MeV, K_{Xe} = 86 MeV

(D) x = n, y = n, K_{Sr} = 86 MeV, K_{Xe} = 129 MeV

Ans. **[A]**

Sol. ${}_{92}U^{236} \longrightarrow {}_{54}Xe^{140} + {}_{38}Sr^{94} + x + y$

(A) Q = (8.5 × 140 + 8.5 × 94 – 7.5 × 236) MeV

= (1989 – 1770) MeV = 219 MeV

then $K_{Xe} + K_{Sr} + k_x + K_y = 219$ MeV

$K_{Xe} + X_{Sr} = 219 - 4 = 215$ MeV

(B) Not satisfying mass conservation and charge conservation.

(C) Not satisfying charge conservation.

(D) Will satisfy all conservation lenz except.

Momentum conservation for D

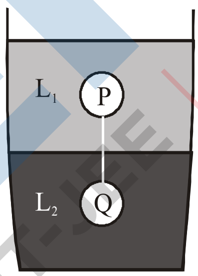
$P_{Xe} = \sqrt{2 \times 140 \times 129} = \sqrt{36120} = 190.05$

$P_{Sr} = \sqrt{2 \times 94 \times 86} = \sqrt{16168} = 127.15$

$P_n = \sqrt{2 \times 1 \times 2} = 2$

Momentum conservation can not be satisfied in last case, so only (A)

11. Two spheres P and Q of equal radii have densities ρ_1 and ρ_2, respectively, The spheres are connected by a massless string and placed in liquids L_1 and L_2 of densities σ_1 and σ_2 and viscosities η_1 and η_2, respectively. They float in equilibrium with the sphere P in L_1 and sphere Q in L_2 and the string being taut (see figure). If sphere P alone in L_2 has terminal velocity $\vec{V}_P$ and Q alone in L_1 has terminal velocity $\vec{V}_Q$, then



(A) $\dfrac{|\vec{V}_P|}{|\vec{V}_Q|} = \dfrac{\eta_1}{\eta_2}$ (B) $\dfrac{|\vec{V}_P|}{|\vec{V}_Q|} = \dfrac{\eta_2}{\eta_1}$ (C) $\vec{V}_P.\vec{V}_Q > 0$ (D) $\vec{V}_P.\vec{V}_Q < 0$

Ans. A, D

Sol. $T + v\rho_1 g = v\sigma_1 g$

$T = v(\sigma_1 - \rho_1)g$ ....(i)

$= v(\rho_2 - \sigma_2)g$ ....(ii)

$\sigma_1 - \rho_1 = \rho_2 - \sigma_2$

$\sigma_1 + \sigma_2 = \rho_1 + \rho_2$

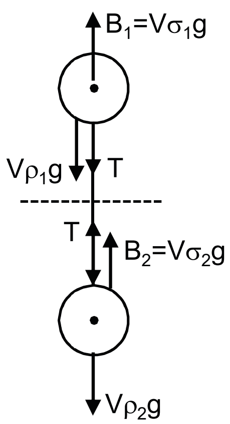
$6\pi\eta_2 r v_P = mg - B = (v\rho_1 g - v\sigma_2 g)$

$= v(\rho_1 - \sigma_2)g$

$6\pi\eta_1 \rho v_Q = mg - B = v(\rho_2 - \sigma_1)g$

So, $6\pi\eta_2 r v_P = -6\pi\eta_1 r v Q$

So, $\dfrac{|v_P|}{|v_Q|} = \dfrac{n_1}{n_2}$ and $\vec{v}_p \bullet \vec{v}_Q < 0$

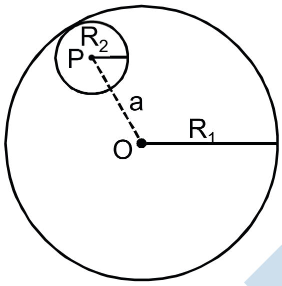


12. In terms of potential difference V, electric current I, permittivity ε_0, permeability μ_0 and speed of light c, the dimensionally correct equations is (are)

(A) $\mu_0 I^2 = \varepsilon_0 V^2$ (B) $\varepsilon_0 I = \mu_0 V$ (C) $I = \varepsilon_0 cV$ (D) $\mu_0 cI = \varepsilon_0 V$

Ans. **[A, C]**

13 Consider a uniform spherical charge distribution of radius R_1 centred at the origin O. In this distribution, a spherical cavity of radius R_2, centred at P with distance OP = a = $R_1 - R_2$ (see figure) is made. If the electric field inside the cavity at position $\vec{r}o = \vec{E}(\vec{r})$. then the correct statement(s) is (are)



(A) $\vec{E}$ is uniform, its magnitude is independent of R_2 but its direction depends on $\vec{r}$

(B) $\vec{E}$ is uniform, its magnitude depends on R_2 and its direction depends on $\vec{r}$

(C) $\vec{E}$ is uniform, its magnitude is independent of a but its direction depends on $\vec{a}$

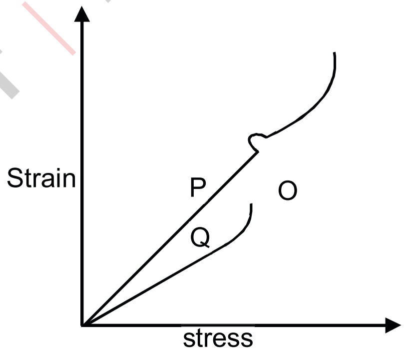
(D) $\vec{E}$ is uniform and both its magnitude and direction depend on $\vec{a}$

Ans. **[D]**

Sol. $\vec{E}_{cavity} = \frac{\rho}{3\varepsilon_0}\vec{a}$ a = $(R_1 - R_2)$

= uniform $\vec{E}$, magnitude and direction depends on $\vec{a}$

14. In plotting stress versus strain curves for two materials P and Q, a student by mistake puts strain on the y-axis and stress on the x-axis as shown in the figure. Then the correct statement(s) is(are)



(A) P has more tensile strength than Q

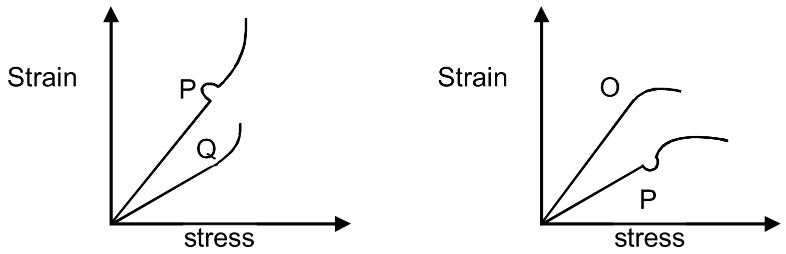
(B) P is more ductile than Q

(C) P is more brittle than Q

(D) The Young's modulus of P is more than that of Q

Ans. **[A, B]**

Sol.



Q has more tensile strength (A)

P is more ductile than Q (B)

Y of Q is more than P, So (D) is incorrect

15. A spherical body of radius R consists of a fluid of constant density and is in equilibrium under its own gravity. If P(r) is the pressure at r(r < R), then the correct option(s) is (are)

(A) $P(r=0)=0$

(B) $\frac{P(r=3R/4)}{P(r=2R/3)}=\frac{63}{80}$

(C) $\frac{P(r=3R/5)}{P(r=2R/5)}=\frac{16}{21}$

(D) $\frac{P(r=R/2)}{P(r=R/3)}=\frac{20}{27}$

Ans. **[B, C]**

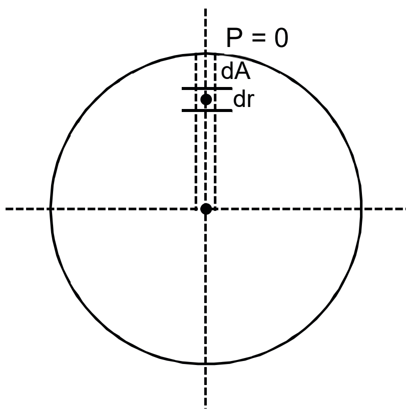
Sol. $g=\frac{GM}{R^3}r$

$dF=-(dm)g=\frac{GM}{R^3}\rho\, dA\, r\, dr$

$dP=-\frac{dF}{dA}=-\frac{GM}{R^3}\rho r dr$

$\int_P^0 dP == -\frac{GM}{R^3}\int_r^R \rho r dr$

$+P=+\frac{GM}{R^3}\frac{\rho}{2}(R^2-r^2)$

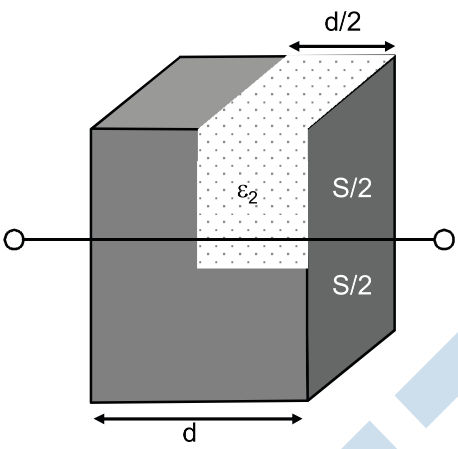


$\frac{P(r=3R/4)}{P(r=2R/3)}=\frac{\left(R^2-(9/16)R^2\right)}{\left(R^2-(4/9)R^2\right)}=\frac{7}{16}\times\frac{9}{5}=\frac{63}{80}$

(C) $\frac{P(r=3R/5)}{P(r=2R/5)}=\frac{(R^2-(9/25)R^2}{(R^2-(4/25)R^2)}=\frac{16}{25}\times\frac{25}{21}=\frac{16}{21}$

(D) $\frac{P(r=R/2)}{P(r=R/2)}=\frac{(R^2-R^2/4)}{(R^2-R^2/9)}=\frac{3/4}{8/9}=\frac{3}{4}\times\frac{9}{8}=\frac{27}{32}$

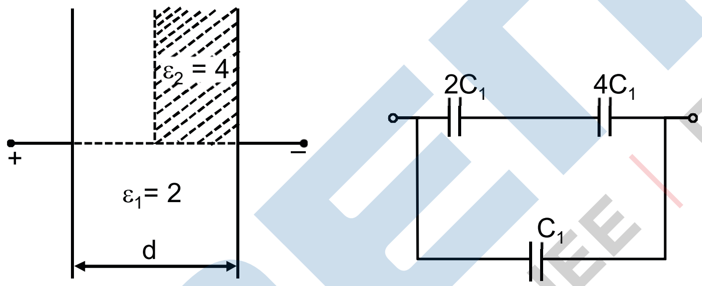
16. A parallel plate capacitor having plates of area S and plate separation d, has capacitance C_1 in air. When two dielectrics of different relative permittivities ($\varepsilon_1 = 2$ and $\varepsilon_2 = 4$) are introduced between the two plates as shown in the figure, the capacitance becomes C_2. The ratio $\frac{C_2}{C_1}$ is



(A) 6/5 (B) 5/3 (C) 7/5 (D) 7/3

Ans. **[D]**

Sol. $C_1 = \frac{\varepsilon_0 S}{d}$



$$C_{eq} = C_1 + \frac{8C_1^2}{6C_1} = C_1 + \frac{4}{3}C_1 = \frac{7}{3}C_1$$

SECTION 3

(Maximum Marks : 16)

- This section contains TWO paragraphs
- Based on each paragraph, there will be TWO questions
- Each question has FOUR options (A), (B), (C) and (D). One or more than one of these four option(s) is(are) correct
- For each question darken the bubble(s) corresponding to all the correct option(s) is the ORS
- Marking scheme:

+ 4 If only the bubble (s) corresponding to all the correct option(s) is/are darkened

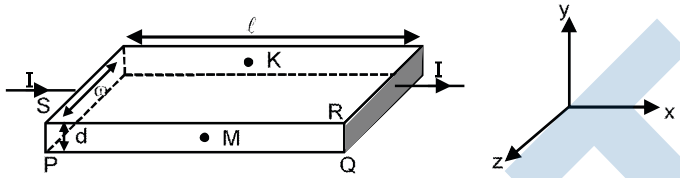
0 If none of the bubbles is darkened

– 2 In all other cases

PARAGRAPH 1

In a thin rectangular metallic strip a constant current I flows along the positive x-direction, as shown in the figure. The length, width and thickness of the strip are l, w and d, respectively.

A uniform magnetic field $\vec{B}$ is applied on the strip along the positive y-direction. Due to this, the charge carriers experience a net deflection along the z-direction. This results in accumulation of charge carriers on the surface PQRS and appearance of equal and opposite charges on the face opposite to PQRS. A potential difference along the z-direction is thus developed. Charge accumulation continues until the magnetic force is balanced by the electric force. The current is assumed to be uniformly distributed on the cross section of the strip and carried by electrons.



17. Consider two different metallic strips (1 and 2) of the same material. Their lengths, are the same, width are w_1 and w_2 and thickness are d_1 and d_2 respectively. Two points K and M are symmetrically located on the opposite faces parallel to the x-y plane (see figure). V_1 and V_2 are the potential differences between K and M in strips 1 and 2, respectively. Then, for a given current I flowing through them in a given magnetic field strength B, the correct statement(s) is (are).

(A) If $w_1 = w_2$ and $d_1 = 2d_2$, then $V_2 = 2V_1$

(B) If $w_1 = w_2$ and $d_1 = 2d_2$, then $V_2 = V_1$

(C) If $w_1 = 2w_2$ and $d_1 = d_2$, then $V_2 = 2V_1$

(D) If $w_1 = 2w_2$ and $d_1 = d_2$, then $V_2 = V_1$

Ans. **[A, D]**

Sol. $\frac{I}{ne\omega d}B = \frac{V}{w}$

$v = \left(\frac{IB}{ned}\right)$

$\frac{v_1}{v_2} = \frac{d_2}{d_1} = \frac{1}{2}$ So, $v_2 = 2v_1$

18. Consider two different metallic strips (1 and 2) of same dimensions (length *l*, width ω and thickness d) with carrier densities n_1 and n_2, respectively. Strip 1 is placed in magnetic field B_1 and strip 2 is placed in magnetic field B_2, both along positive y-directions. Then V_1 and V_2 are the potential differences developed between K and M is strips 1 and 2, respectively. Assuming that the current I is the same for both the strips, the correct option(s) is (are)

(A) If $B_1 = B_2$ and $n_1 = 2n_2$, then $V_2 = 2V_1$

(B) If $B_1 = B_2$ and $n_1 = 2n_2$, then $V_2 = V_1$

(C) If $B_1 = 2B_2$ and $n_1 = n_2$, then $V_2 = 0.5V_1$

(D) If $B_1 = 2B_2$ and $n_1 = n_2$, then $V_2 = V_1$

Ans. **[A, C]**

Sol. $ev_{d_1}B_1 = e\frac{v_1}{\omega}$

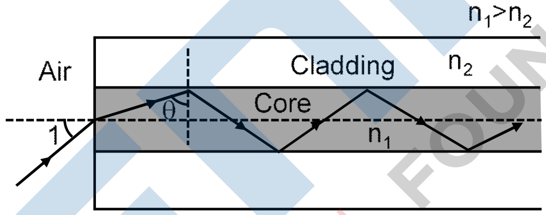
$\Rightarrow v_1 = \frac{\omega IB_1}{n_1 ewd} = \frac{IB_1}{n_1 de}$

$\frac{v_1}{v_2} = \frac{B_1}{n_1}\frac{n_2}{B_2} = \frac{1}{2} = \ = \qquad \Rightarrow v_2 = 2v_1$

$\frac{v_1}{v_2} = 2 \qquad \Rightarrow v_2 = 0.5\, v_1$

PARAGRAPH 2

Light guidance in an optical fiber can be understood by considering a structure comprising of thin solid glass cylinder of refractive index n_1 surrounded by a medium of lower refractive index n_2. The light guidance in the structure takes place due to successive total internal reflections at the interface of the media n_1 and n_2 as shown in the figure. All rays with the angle of incidence i less than a particular value i_m are confined in the medium of refractive index n_1. The numerical aperture (NA) of the structure is defined as sin i_m.



19. For two structures namely S_1 with $n_1 = \sqrt{45}/4$ and $n_2 = 3/2$, and S_2 with $n_1 = 8/5$ and $n_2 = 7/5$ and taking the refractive index of water to be 4/3 and that of air to be 1, the correct option(s) is(are)

(A) NA of S_1 immersed in water is the same as that of S_2 immersed in a liquid of refractive index $\frac{16}{3\sqrt{15}}$

(B) NA of S_1 immersed in liquid of refractive index $\frac{16}{\sqrt{15}}$ is the same as that of S_2 immersed in water.

(C) NA of S_1 placed in air is the same as that of S_2 immersed in liquid of refractive index $\frac{4}{\sqrt{15}}$.

(D) NA of S_1 placed in air is the same as that of S_2 placed in water.

Ans. **[A, C]**

Sol. $\sin C = \frac{\mu_r}{\mu_d} = \frac{n_2}{n_1}$

$\mu_s \sin i = n \sin r = n_1 \sin (90 - Q)$

$\mu_s \sin i = n_1 \cos Q = n1\sqrt{1 - \frac{n_2^2}{n_1^2}} = \sqrt{n_1^2 - n_2^2}$

$\sin i = \frac{1}{\mu_s}\sqrt{n_1^2 - n_2^2}$

(A) $S_1 : \sin i_{m_1} = \frac{3}{4}\sqrt{\frac{45}{16} - \frac{9}{4}} = \frac{3}{4}\sqrt{\frac{45-36}{16}} = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$

$S_2 : \sin i_{m_2} = \frac{3\sqrt{15}}{16}\sqrt{\frac{64}{25} - \frac{49}{25}} = \frac{3\sqrt{15}}{16}\frac{\sqrt{15}}{5} = \frac{9}{16}$

(B) $S_1 : \sin i_{m_1} = \frac{\sqrt{15}}{6}\sqrt{\frac{45}{16} - \frac{9}{4}} = \frac{\sqrt{15}}{6}\sqrt{\frac{45-36}{16}} = \frac{3}{4}\frac{\sqrt{15}}{6} = \frac{\sqrt{15}}{8}$

$S_2 : \sin i_{m_2} = \frac{3}{4}\sqrt{\frac{64}{25} - \frac{49}{25}} = \frac{3}{4}\sqrt{\frac{15}{25}} = \frac{\sqrt{15} \times 3}{20}$

(C) $S_1 : \sin i_{m_1} = \sqrt{\frac{45}{16} - \frac{9}{4}} = \frac{\sqrt{9}}{4} = \frac{3}{4}$

$S_2 : \sin i_{m_2} = \frac{\sqrt{15}}{4}\sqrt{\frac{15}{25}} = \frac{15}{4 \times 5} = \frac{15}{20} = \frac{3}{4}$

(D) $S_1 : \sin i_{m_1} = \frac{3}{4}$

$S_2 : \sin i_{m_2} = \frac{\sqrt{15}}{5}$

20. If two structures of same cross-sectional area, but different numerical apertures NA_1 and NA_2 ($NA_2 < NA_1$) are joined longitudinally, the numerical aperture of the combined structure is

(A) $\frac{NA_1 NA_2}{NA_1 + NA_2}$ (B) $NA_1 + NA_2$ (C) NA_1 (D) NA_2

Ans. **[D]**

PART B: CHEMISTRY

(Maximum Marks : 32)

- This section contains Eight questions.
- The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive.
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- Marking Scheme

  + 4 If the bubble corresponding to the answer is darkened.

  0 In all other cases.

21. The molar conductivity of a solution of a weak acid HX (0.01 M) is 10 times smaller than the molar conductivity of a solution of a weak acid HY (0.10 M). If $\lambda^0_{X^-} \approx \lambda^0_{Y^-}$, the difference in their pK_a values, $pK_a(HX) - pK_a(HY)$, is (consider degree of ionisation of both acids to be << 1)

Ans. **[3]**

Sol. HX ⇌ $H^+ + X^-$

$$Ka = \frac{[H^+][X^-]}{[HX]}$$

HX ⇌ $H^+ + Y^-$

$$Ka = \frac{[H^+][Y^-]}{[HY]}$$

$$\Lambda_m \text{for HX} = \Lambda_{m_1}$$

$$\Lambda_m \text{for HY} = \Lambda_{m_2}$$

$$\Lambda_{m_1} = \frac{1}{10}\Lambda_{m_2}$$

$Ka = C\alpha^2$

$$Ka_1 = C_1 \times \left(\frac{\Lambda_{m_1}}{\Lambda^0_{m_1}}\right)^2$$

$$Ka_2 = C_2 \times \left(\frac{\Lambda_{m_2}}{\Lambda^0_{m_2}}\right)^2$$

$$\frac{Ka_1}{Ka_2} = \frac{C_1}{C_2} \times \left(\frac{\Lambda_{m_1}}{\Lambda^0_{m_2}}\right)^2 = \frac{0.01}{0.1} \times \left(\frac{1}{10}\right)^2 = 0.001$$

$pKa_1 - pKa_2 = 3$

22. A closed vessel with rigid walls contains 1 mol of $^{238}_{92}U$ and 1 mol of air at 298 K. Considering complete decay of $^{238}_{92}U$ to $^{206}_{82}Pb$ to , the ratio of the final pressure to the initial pressure of the system at 298K is

Ans. **[9]**

Sol. In conversion of $^{238}_{92}U$ to $^{206}_{82}Pb$, 8 α - particles and 6 β particles are ejected.

The number of gaseous moles initially = 1 mol

The number of gaseous moles finally = 1 + 8 mol; (1 mol from air and 8 mol of $_2He^4$)

So the ratio = 9/1 = 9

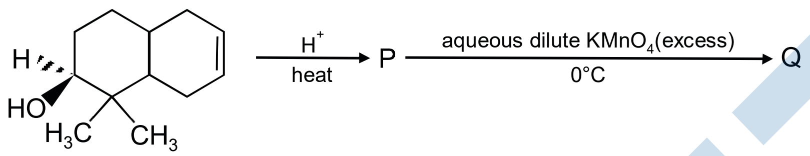
23. In dilute aqueous H_2SO_4, the complex diaquodioxalatoferrate(II) is oxidized by MnO_4^- .For this reaction, the ratio of the rate of change of $[H^+]$ to the rate of change of $[MnO_4^-]$ is

Ans. **[8]**

Sol. $$[Fe(C_2O_4)(H_2O)]^{2-} + MnO_4^{2-} + 8H^+ \longrightarrow Mn^{2+} + Fe^{3+} + Fe^{3+} + 4CO_2 + 6H_2O$$

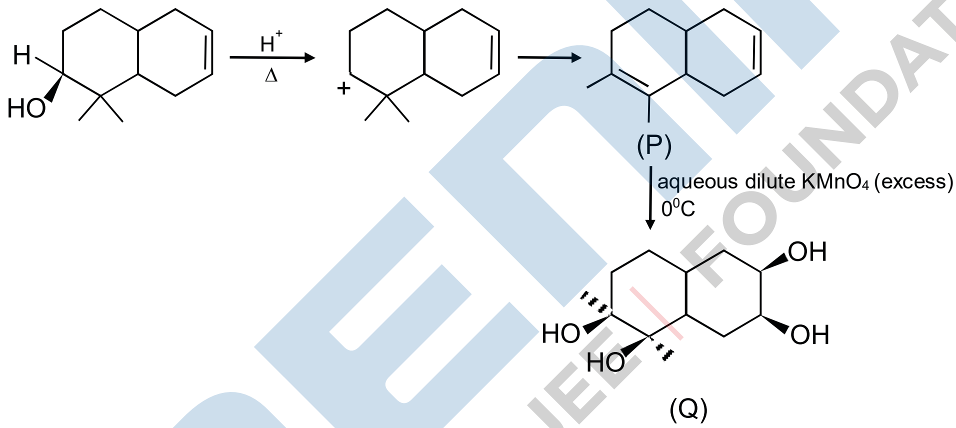
So the ratio of rate of change of $[H^+]$to that of rate$\left[MnO_4^-\right]$of is 8.

24. The number of hydroxyl group(s) in **Q** is

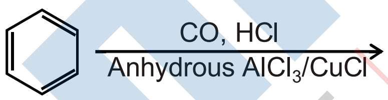


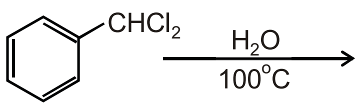
Ans. **[4]**

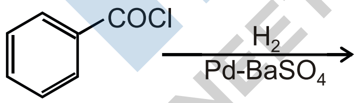
Sol.

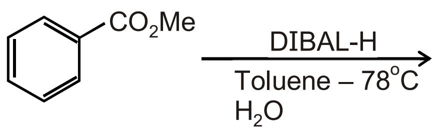


25. Among the following, the number of reaction(s) that produce(s) benzaldehyde is :

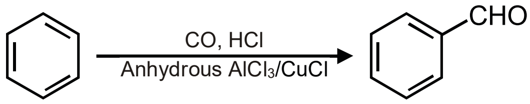
I. 

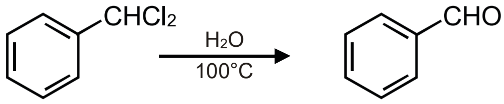
II. 

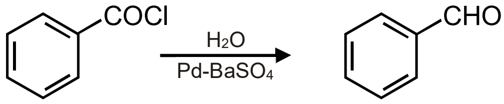
III. 

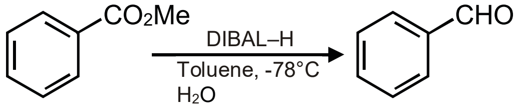
IV. 

Ans. **[4]**

Sol. I 

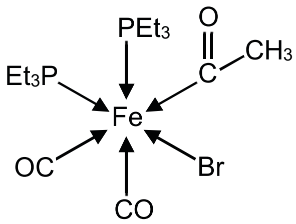
II 

III 

IV 

26. In the complex acetylbromidodicarbonylbis(triethylphosphine)iron(II), the number of Fe–C bond(s) is

Ans. **[3]**

Sol. 

The number of – C bonds is 3.

27. Among the complex ions, $[Co(NH_2-CH_2-CH_2-NH_2)_2Cl_2]^+$, $[CrCl_2(C_2O_4)_2]^{3-}$, $[Fe(H_2O)_4(OH)_2]^+$, $[Fe(NH_3)_2(CN)_4]^-$, $[Co(NH_2-CH_2-CH_2-NH_2)_2(NH_3)Cl]^{2+}$ and $[Co(NH_3)_4(H_2O)Cl]^{2+}$, the number of complex ion(s) that show(s) cis-trans isomerism is :

Ans. **[6]**

Sol. $[CO(en)_2Cl_2]^+ \longrightarrow$ will show cis – trans isomerism

$[CrCl_2(C_2O_4)_2]^{3-} \longrightarrow$ will show cis - trans isomerism

$[Fe(H_2O)_4(OH)_2]^- \longrightarrow$ will show cis trans isomerism

$[Fe(CN)_4(NH_3)_2]^- \longrightarrow$ will show cis trans isomerism

$[Co(en)_2(NH_3)Cl]^{2+} \longrightarrow$ will show cis trans isomerism

$[Co(NH_3)_4(H_2O)Cl^{2+}] \longrightarrow$ will not show cis– trans isomerism (Although it will show geometrical isomerism)

28. Three moles of B_2H_6 are completely reacted with methanol. The number of moles of boron containing product formed is:

Ans. **[6]**

Sol. $B_2H_6 + 6MeOH \longrightarrow 2B(OMe)_3 + 6H_2$

1 mole of B_2H_6 reacts with 6 mole of MeOH to give 2 moles of $B(OMe)_3$.

3 mole of B_2H_6 will react with 18 mole of MeOH to give 6 moles of $B(OMe)_3$

SECTION 2

(Maximum Marks : 32)

- This section contains EIGHT questions
- Each question has FOUR options (A), (B), (C) and (D). ONE OR MORE THAN ONE of these four option(s) is(are) correct
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS
- Marking scheme:

  +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened

  0 If none of the bubbles is darkened

  –2 In all other cases

29. When O_2 is adsorbed on a metallic surface, electron transfer occurs from the metal to O_2. The **TRUE** statement(s) regarding this adsorption is(are)

(A) O_2 is physisorbed (B) heat is released

(C) occupancy of π^*_{2p} of O_2 is increased (D) bond length of O_2 is increased.

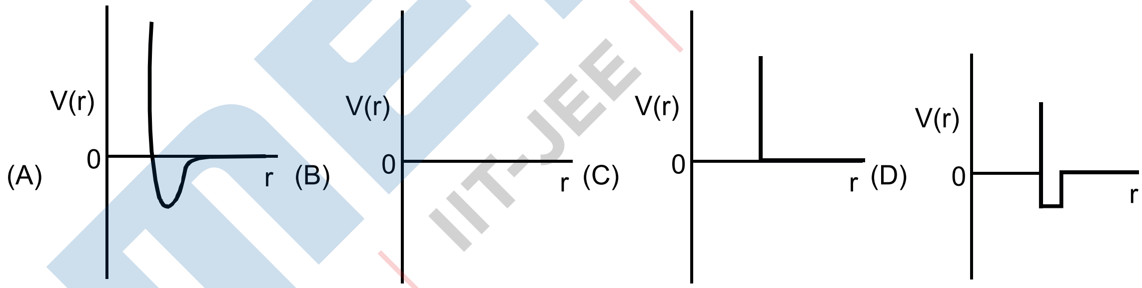
Ans. **[B], [C], [D]**

Sol. * Adsorption of O_2 on metal surface is exothermic.

* During electron transfer from metal to O_2 electron occupies π^*_{2p} orbital of O_2.

* Due to electron transfer to O_2 the bond order of O_2 decreases hence bond length increases.

30. One mole of a monoatomic real gas satisfies the equation p(V – b) = RT where b is a constant. The relationship of interatomic potential V(r) and interatomic distance r for the gas is given by:

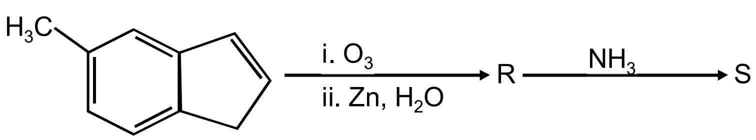


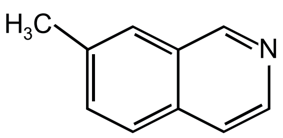
Ans. **[C]**

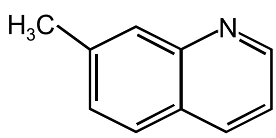
Sol. At large inter-ionic distances (because a → 0) the P.E. would remain constant.

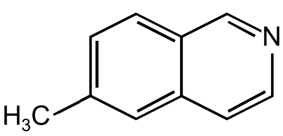
However, when r → 0; repulsion would suddenly increase

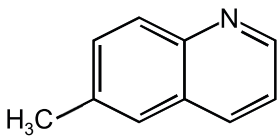
31. In the following reactions, the product **S** is



(A)


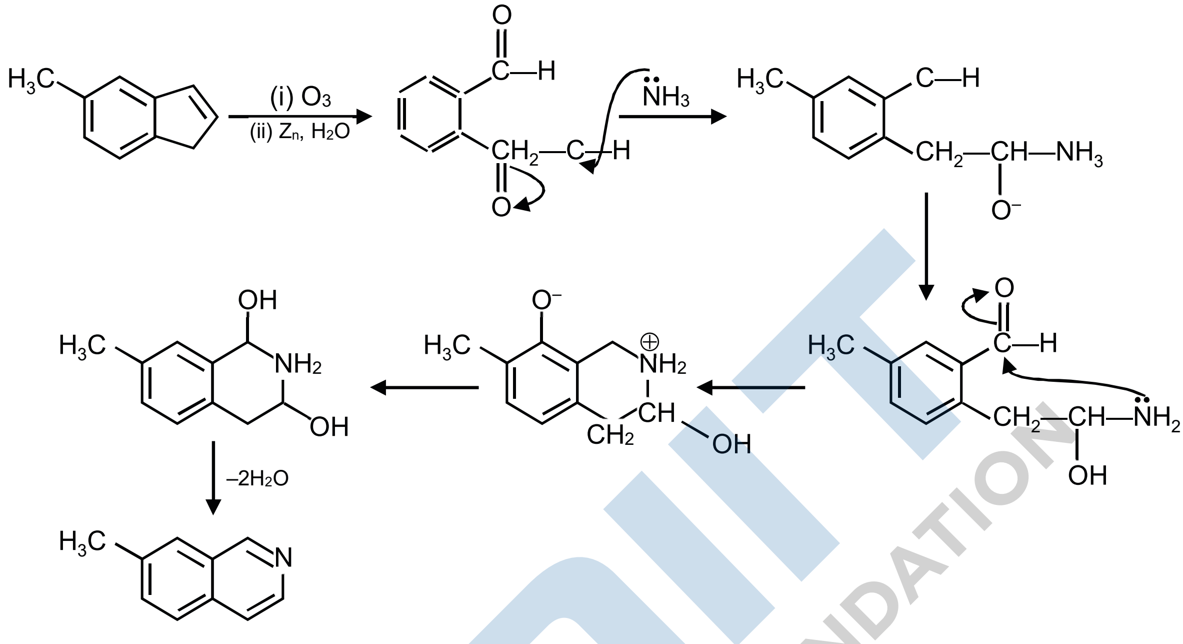
(B)


(C) 

(D) 

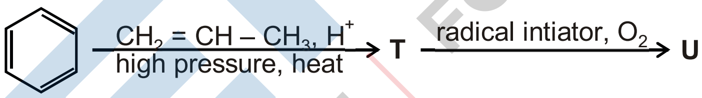
Ans. [A]

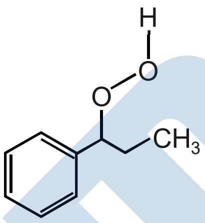
Sol.

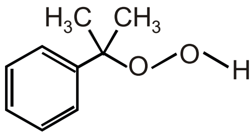


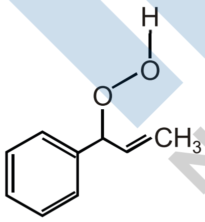
(S)

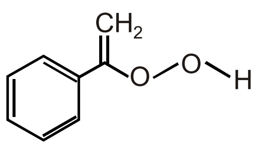
32. The major product **U** in the following reactions is



(A) 

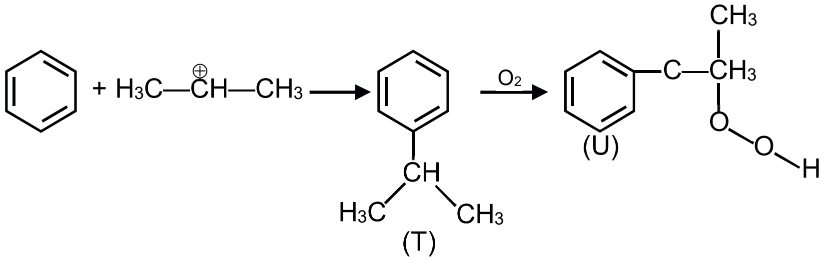
(B) 

(C) 

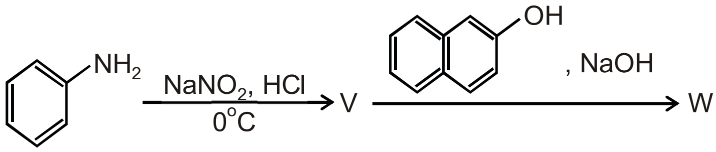
(D) 

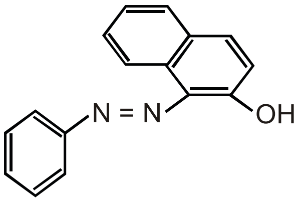
Ans. **[B]**

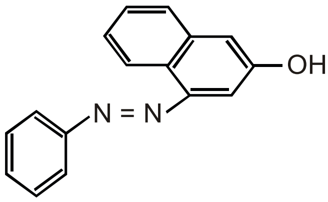
Sol.

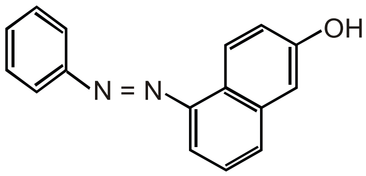


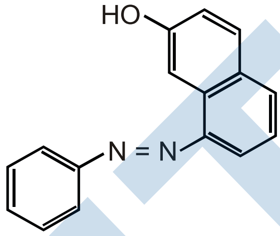
33 In the following reactions, the major product **W** is



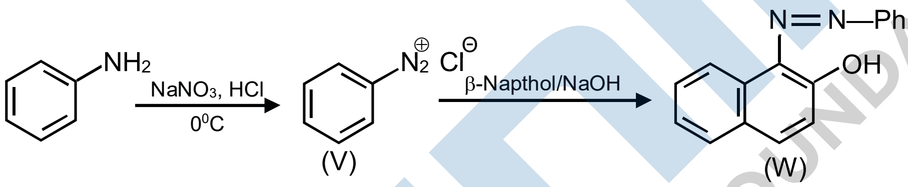
(A)


(B)


(C)


(D)


Ans. **[A]**

Sol.


34. The correct statement(s) regarding, (i) HClO, (ii) $HClO_2$, (iii) $HClO_3$ and (iv) $HClO_4$, is(are)

(A) The number of Cl = O bonds in (ii) and (iii) together is two

(B) The number of lone pairs of electrons on Cl in (ii) and (iii) together is three

(C) The hybridization of Cl in (iv) is sp^3

(D) Amongst (i) to (iv), the strongest acid is (i)

Ans. **[B], [C]**

Sol.

H—O—Cl (I)

H—O—C̈l=0 (II)

H—O—C̈l=0 (III)
‖
O

O
‖
H—O—Cl=0 (IV)
‖
O

35. The pair(s) of ions where BOTH the ions are precipitated upon passing H_2S gas in presence of dilute HCl, is(are)

(A) Ba^{2+}, Zn^{2+} (B) Bi^{3+}, Fe^{3+} (C) Cu^{2+}, Pb^{2+} (D) Hg^{2+}, Bi^{3+}

Ans. **[C], [D]**

Sol. Cu^{2+}, Pb^{2+}, Hg^{2+}, Bi^{3+} give ppt with H_2S in presence of dilute HCl.

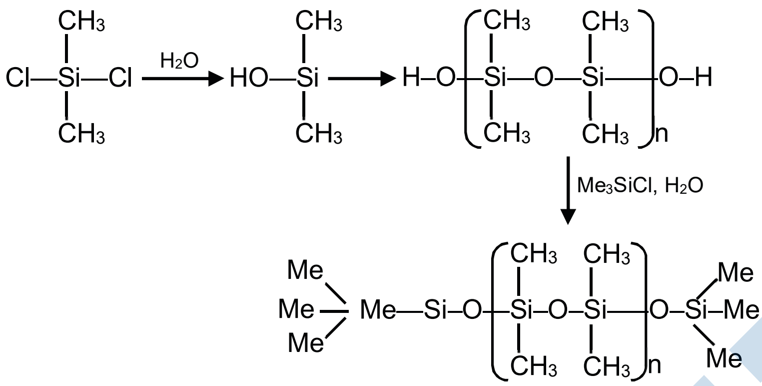
36. Under hydrolytic conditions, the compounds used for preparation of linear polymer and for chain termination, respectively, are

(A) CH_3SiCl_3 and $Si(CH_3)_4$ (B) $(CH_3)_2SiCl_2$ and $(CH_3)_3SiCl$

© $(CH_3)_2SiCl_2$ and CH_3SiCl_3 (D) $SiCl_4$ and $(CH_3)_3SiCl$

Ans. [B]

Sol.



SECTION 3

(Maximum Marks : 16)

. This section contains TWO paragraphs

. Based on each paragraph, there will be TWO questions

. Each question has FOUR options (A), (B), (C) and (D). One or more than one of these four option(s) is(are) correct

. For each question darken the bubble(s) corresponding to all the correct option(s) in the ORS

. Marking scheme:

+ 4 If only the bubble (s) corresponding to all the correct option(s) is/are darkened

0 If none of the bubbles is darkened

– 2 In all other cases

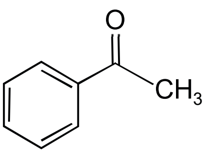
PARAGRAPH - 1

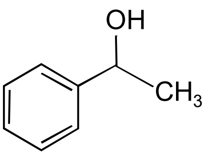
In the following reactions:

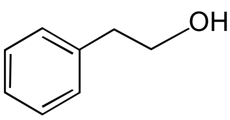
$$C_8H_6 \xrightarrow[H_2]{Pd-BaSO_4} C_8H_8 \xrightarrow[ii.\ H_2O_2,\ NaOH,\ H_2O]{i.\ B_2H_6} X$$

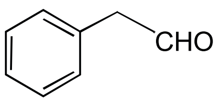
$$C_8H_6 \xrightarrow{H_2O,\ HgSO_4,\ H_2SO_4} C_8H_8O \xrightarrow[ii.\ H^+,\ Heat]{i.\ EtMgBr,\ H_2O} Y$$

37. Compound X is

(A) 

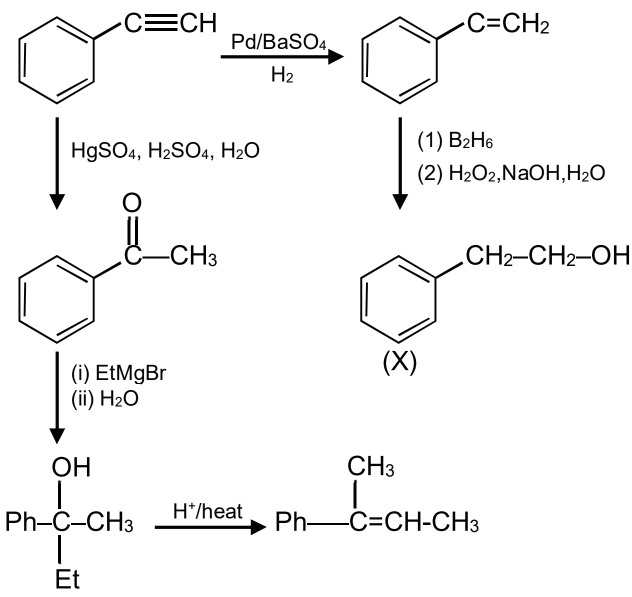
(B) 

(C) 

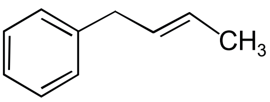
(D) 

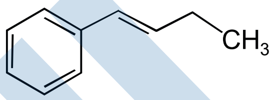
Ans. [C]

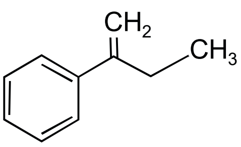
Sol.

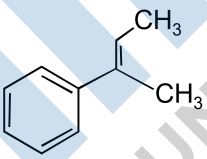


38. The major compound Y is

(A) 

(B) 

(C) 

(D) 

Ans. **[D]**

PARAGRAPH - 2

When 100 mL of 1.0 M HCl was mixed with 100 mL of 1.0 M NaOH in an insulated beaker at constant pressure, a temperature increase of 5.7°C was measured for the beaker and its contents (**Expt. 1**). Because the enthalpy of neutralization of a strong acid with a strong base is a constant ($-$ 57.0 kJ mol^{-1}), this experiment could be used to measure the calorimeter constant.

In a second experiment (**Expt. 2**), 100 mL of 2.0 M acetic acid (K_a = 2.0 × 10^{-5}) was mixed with 100 mL of 1.0 M NaOH (under identical conditions of **Expt. 1**) where a temperature rise of 5.6°C was measured.

(Consider heat capacity of all solutions as 4.2 J $g^{-1}K^{-1}$ and density of all solutions as 1.0 g mL^{-1})

39. Enthalpy of dissociation **(in kJ mol^{-1})** of acetic acid obtained from the **Expt. 2** is

(A) 1.0 (B) 10.0 (C) 24.5 (D) 51.4

Ans. **[A]**

Sol. HCl + NaOH $\longrightarrow$ NaCl + H_2O

n = 100 ×1 = 100 m mole = 0.1 mole

Energy evolved due to neutralization of HCl and NaOH = 0.1 × 57 = 5.7 kJ = 5700 Joule

Energy used to increase temperature of calorimeter = 5700 – 4788 = 912 Joule

ms. Δt = 912

m.s × 5.7 = 912

ms = 160 Joule/°C [Calorimeter constant]

Energy evolved by neutralization of CH_3COOH and NaOH

= 200× 4.2×5.6 +160×5.6 = 5600 Joule

So energy used in dissociation of 0.1 mole CH3COOH = 5700 □ 5600 = 100 Joule

Enthalpy of dissociation = 1 kJ/mole

40. The pH of the solution after **Expt.2** is

(A) 2.8 (B) 4.7 (C) 5.0 (D) 7.0

Ans. **[B]**

Sol. $CH_3COOH = \frac{1 \times 100}{200} = \frac{1}{2}$

$CH_3CONa = \frac{1 \times 100}{200} = \frac{1}{2}$

$pH = pK_a + \log\frac{[salt]}{[acid]}$

$pH = 5 - \log 2 + \log\frac{1/2}{1/2}$

pH=4.7

PART C: MATHEMATICS

Section 1 (Maximum Marks : 32)

- This section contains Eight questions.
- The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive.
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- Marking Scheme

  + 4 If the bubble corresponding to the answer is darkened.

  0 In all other cases.

41. If $\alpha = \int_0^1 (e^{9x+3\tan^{-1}x})\left(\frac{12+9x^2}{1+x^2}\right)dx$ where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is

Ans. 9

Sol. $\alpha = \int_0^1 (e^{9x+3\tan^{-1}x})\left(\frac{12+9x^2}{1+x^2}\right)dx$

Let $9x + 3\tan^{-1}x = t$

$\Rightarrow \left(9+\frac{3}{1+x^2}\right)dx = dt$

$\alpha = \int_0^{\left(9+\frac{3\pi}{4}\right)} e^t\,dt = \left(e^t\right)_0^{\left(9+\frac{3\pi}{4}\right)} = e^{\left(9+\frac{3\pi}{4}\right)} - 1$

$\therefore 1+\alpha = e^{\left(9+\frac{3\pi}{4}\right)}$

$\Rightarrow \log_e|1+\alpha| = 9 + \frac{3\pi}{4}$

$\Rightarrow \log_e|1+\alpha| - \frac{3\pi}{4} = 9$. **Ans.**

42. Let $f : R \rightarrow R$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$.

Suppose that $F(x) = \int_{-1}^{x} f(t)\,dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t\left|f(f(t))\right|dt$ for all $x \in [-1, 2]$.

If $= \lim_{x\to1}\frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is

Ans. 7

Sol. $f(-x) = -f(x)$

$\therefore f(0) = 0$ and $f(1) = \frac{1}{2}$

$F(x) = \int_{-1}^{x} f(t)\,dt = F'(x) = f(x)$

$$G(x) = \int_{-1}^{x} t\left|f(f(t))\right|dt \Rightarrow G'(x) = x\,|f(f(x))|$$

$$\lim_{x\to 1}\frac{F(x)}{G(x)} = \lim_{x\to 1}\frac{F'(x)}{G'(x)} = \frac{F'(1)}{G'(1)} = \frac{f(1)}{\left|f(f(1))\right|} = \frac{\frac{1}{2}}{\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14}$$

$$\Rightarrow \left|f\left(\frac{1}{2}\right)\right| = 7$$

f(x) vanishes at only one point

$\therefore \ f\left(\frac{1}{2}\right) = 7$. **Ans.**

43. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R^3. Let the components of a vector $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ be 4, 3 and 5 respectively. If the components of this vector $\vec{s}$ along $(-\vec{p}+\vec{q}+\vec{r}), (\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}, -\vec{q}+\vec{r})$ are x, y and z respectively, then the value of 2x + y + z $\dfrac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|}$ is

Ans. **Bonus**

44. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where i = $i = \sqrt{-1}$. The value of the expression is

Ans. **4**

Sol. $\alpha_k = e^{\frac{ik\pi}{7}} = \alpha^k$ where $\alpha = e^{\frac{i\pi}{7}}$ and $|\alpha| = 1$

$$\frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|} = \frac{\sum_{k=1}^{12}|\alpha|^k|\alpha-1|}{\sum_{k=1}^{3}|\alpha|^{4k-2}|\alpha-1|} = \frac{\sum_{k=1}^{12}|\alpha-1|}{\sum_{k=1}^{3}|\alpha-1|} = \frac{12|\alpha-1|}{3|\alpha-1|} = 4.$$ **Ans.**

45. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6: 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is

Ans. **9**

Sol. $$\because \frac{S_7}{S_{11}} = \frac{\frac{7}{2}(2a+6d)}{\frac{11}{2}(2a+10d)} = \frac{6}{11}$$

$\Rightarrow$ 7 (2a + 6d) = 6 (2a + 10d)

$\Rightarrow$ 2a = 18d $\Rightarrow$ a = 9d

T_7 = a + 6d

$130 < a + 6d < 140 \Rightarrow 130 < 15d < 140$

$\Rightarrow \frac{130}{15} < d < \frac{140}{15} \Rightarrow \frac{26}{3} < d < \frac{28}{3}$

$\therefore$ d = 9. **Ans.**

46. The Coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3)......(1 + x^{100})$ is

Ans. 8

Sol. x^9 can be obtained by multiplying terms containing powers of x.

(0, 9), (1, 8), (2, 7), (3, 6), (4, 5), (1, 2, 6), (1, 3, 5), (2, 3, 4)

$\therefore$ coefficient of x^9 is 8. **Ans.**]

47. Suppose that the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ are $(f_1, 0)$ and $(f_2, 0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2} + m_2^2\right)$ is

Ans. 4

Sol. $e = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$

$\therefore$ foci are (±2, 0)

f_1: (2, 0), f_2: (–2, 0)

Tangent to p_1

$y = m_1x + \frac{2}{m_1}$, which passes through (– 4, 0)

$0 = -4m_1 + \frac{2}{m_1} \Rightarrow m_1^2 = \frac{1}{2}$

Tangent to p_2

$y = m_2x - \frac{4}{m_2}$, which passes through (2, 0)

$0 = 2m_2 - \frac{4}{m_2} \Rightarrow m_2^2 = 2$

$\therefore \frac{1}{m_1^2} + m_2^2 = 2 + 2 = 4$ **Ans.**

48. Let m and n be two positive integers greater than 1. If $\lim_{\alpha \to 0} \left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = -\left(\frac{e}{2}\right)$ then the value of $\frac{m}{n}$ is

Ans. 2

Sol. $\lim_{\alpha\to 0}\frac{e^{\cos(\alpha^n)}-e}{\alpha^m}=-\frac{e}{2}$

$\lim_{\alpha\to 0}\frac{e^{\cos(\alpha^n)-1}-1}{\alpha^m}=\frac{-1}{2}$

$\lim_{\alpha\to 0}\frac{\cos(\alpha^n)-1}{\alpha^m}=\frac{-1}{2}$

$\lim_{\alpha\to 0}\frac{1-\cos(\alpha^n)}{\alpha^{2n}}\cdot\frac{\alpha^{2n}}{\alpha^m}=\frac{-1}{2}$

$\lim_{\alpha\to 0}\frac{1}{2}\cdot\alpha^{2n-m}=\frac{1}{2}$

$2n - m = 0 \Rightarrow \frac{m}{n} = 2$ **Ans.**

SECTION 2

(Maximum Marks : 32)

- This section contains EIGHT questions
- Each question has FOUR options (A), (B), (C) and (D). ONE OR MORE THAN ONE of these four option(s) is(are) correct
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS
- Marking scheme:

  +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened

  0 If none of the bubbles is darkened

  –2 In all other cases

49. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then the correct expression(s) is(are)

(A) $\int_0^{\pi/4} x f(x)dx = \frac{1}{12}$

(B) $\int_0^{\pi/4} f(x)dx = 0$

(C) $\int_0^{\pi/4} x\, f(x)dx = \frac{1}{6}$

(D) $\int_0^{\pi/4} f(x)dx = 1$

Ans. **AB**

Sol. $\sqrt{f(x)} = 7\tan^6 x(\tan^2 x + 1) - 3\tan^2 x(\tan^2 x + 1) = (7\tan^6 x - 3\tan^2 x)\sec^2 x$

$\int_0^{\pi/4} f(x)dx=\int_0^{\pi/4}(7\tan^6 x-3\tan^2 x)\sec^2 x\,dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\int_0^1 (7t^6-3t^2)dt = 0$

$\int_0^{\pi/4} x f(x)dx=\int_0^{\pi/4}\underset{I}{x}\,\underbrace{(7\tan^6 x-3\tan^2 x)\sec^2 x}_{II}\,dx$

$$= \left(x\cdot(\tan^7 x - \tan^3)\right)_0^{\pi/4} - \int_0^{\pi/4} (\tan^7 x - \tan^3 x)dx$$

$$= 0 - \int_0^{\pi/4} \tan^3 x(\tan^4 x - 1)dx$$

$$= -\int_0^{\pi/4} \tan^3 x(\tan^2 x - 1)\sec^2 x\,dx$$

tan x = t

$\sec^2 x\,dx = dt$

$$= -\int_0^1 t^3(t^2-1)dt = -\left(\frac{1}{6}-\frac{1}{4}\right) = \frac{1}{12}$$

50. Let $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)dx \le M$, then the possible values of m and M are

(A) m = 13, M = 24

(B) $m = \frac{1}{4}$, $M = \frac{1}{2}$

(C) m = – 11, M = 0

(D) m = 1, M = 12

Ans. D

Sol. $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}\ \forall x \in R$

Clearly f '(x) is increasing in $\left[\frac{1}{2}, 1\right]$

$f'(x) \le f'(1) = 96$

$f'(x) \ge f'\left(\frac{1}{2}\right) = 8$

for maximum, $f(x) = y = 96\left(x - \frac{1}{2}\right)$

for minimum, $f(x) = y = 8\left(x - \frac{1}{2}\right)$

$$\int_{1/2}^{1} f(x)dx \le \int_{1/2}^{1} 96\left(x-\frac{1}{2}\right)dx = 12$$

$$= \int_{1/2}^{1} f(x)dx \ge \int_{1/2}^{1} 8\left(x-\frac{1}{2}\right)dx = 4x^2 - 4x\Big|_{1/2}^{1} = 0 - (1-2) = 1$$

$$\therefore 1 \le \int_{1/2}^{1} f(x)dx \le 12$$

51. Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals is(are) a subset(s) of S?

(A) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$

(B) $\left(-\frac{1}{\sqrt{5}}, 0\right)$

(C) $\left(0, \frac{1}{\sqrt{5}}\right)$

(D) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

Ans. AD

Sol. $\alpha x^2 - x + \alpha = 0$

$D > 0$

$1 - 4\alpha^2 > 0$

$4\alpha^2 - 1 < 0$

$\alpha \in \left(\frac{-1}{2}, \frac{1}{2}\right)$

$|x_1 - x_2| < 1$

$(x_1 - x_2)^2 < 1$

$(x_1 + x_2)^2 - 4x_1x_2 < 1$

$\left(\frac{1}{\alpha}\right)^2 - 4 < 1$

$\frac{1}{\alpha^2} - 5 < 0$

$\frac{5\alpha^2 - 1}{\alpha^2} > 0$

$5\alpha^2 - 1 > 0, \alpha \neq 0$

$\therefore \alpha \in \left(\frac{-1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) - \{0\}$

$\therefore \alpha \in \left(-\frac{1}{2}, \frac{-1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{3}}, \frac{1}{2}\right)$

52. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are)

(A) $\cos\beta > 0$ (B) $\sin\beta < 0$ (C) $\cos(\alpha + \beta) > 0$ (D) $\cos\alpha < 0$

Ans. BCD

Sol. $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$

$\sin^{-1}\left(\frac{1}{2}\right) < \sin^{-1}\left(\frac{6}{11}\right) < \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$3\left(\frac{\pi}{6}\right) < 3\sin^{-1}\left(\frac{6}{11}\right) < 3\left(\frac{\pi}{4}\right)$

$\frac{\pi}{2} < \alpha < \frac{3\pi}{4}$

$\therefore \alpha \in$ IInd quadrant

$\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$

$$\cos^{-1}\left(\frac{1}{2}\right) < \cos^{-1}\left(\frac{4}{9}\right) < \cos^{-1}\left(\frac{\sqrt{6}-\sqrt{2}}{8}\right)$$

$$3\left(\frac{\pi}{3}\right) < b < 3\frac{5\pi}{12}$$

$$\pi < \beta < \frac{5\pi}{4} \Rightarrow \beta \in \text{III}^{rd} \text{ quadrant}$$

53. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y-1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P, Q and R, respectively. Suppose that PQ = PR = $\frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is (are)

(A) $e_1^2 + e_2^2 = \frac{43}{40}$ (B) $e_1e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$ (C) $\left|e_1^2 - e_2^2\right| = \frac{5}{8}$ (D) $e_1e_2 = \frac{\sqrt{3}}{4}$

Ans. **AB**

Sol. $x - y + \lambda = 0$ normal to circle.

passes centre (0, 1) $\Rightarrow \lambda = 1$

$x - y + 1 = 0$

$x + y = 3$

$2x = 2$

$x = 1, y = 2.$ P(1, 2)

$$\frac{x-1}{\cos 135^\circ} = \frac{y-2}{\sin 135^\circ} = \pm\frac{2\sqrt{2}}{3}$$

$x = 1 \pm$ $x = 1 \pm \left(\frac{-2}{3}\right) = \frac{1}{3}, \frac{5}{3};$ $y = 2 \pm \left(\frac{2}{3}\right) = \frac{8}{3}, \frac{4}{3}$

$$Q\left(\frac{5}{3}, \frac{4}{3}\right), R\left(\frac{1}{3}, \frac{8}{3}\right)$$

$$E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$$

$$T_Q : \frac{5x}{3a^2} + \frac{4}{3}\frac{y}{b^2} = 1$$

Compare with $x + y = 3$

$$\frac{5}{3a^2} = \frac{4}{3b^2} = \frac{1}{3}$$

$a^2 = 5, b^2 = 4$

$$= e_1^2 = 1 - = \frac{b^2}{a^2} = \frac{1}{5}$$

$$E_2 : \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \quad (A < B)$$

$T_R : \frac{1}{3A^2}x + \frac{8}{3B^2} = 1$

Compare with x + y = 3

$\frac{1}{3A^2} = \frac{8}{3B^2} = \frac{1}{3}$

$A^2 = 1$

$B^2 = 8$

$e_2^2 = 1 - \frac{1}{8} = \frac{7}{8}$

$e_1^2 + e_2^2 = \frac{8+35}{40} = \frac{43}{40}$

$\Rightarrow e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$

54. Consider the hyperbola H : $x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (ℓ, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are)

(A) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$

(B) $\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2 - 1}\right)}$ for $x_1 > 1$

(C) $\frac{d\ell}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$

(D) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

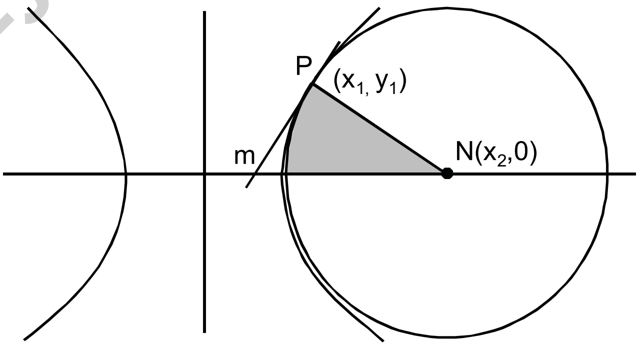
Sol **ABD**

Sol. H : $x^2 - y^2 = 1$

Tangent at P (x_1, y_1)

$xx_1 - yy_1 = 1$

$M \equiv \left(\frac{1}{x_1}, 0\right)$



Centroid of the triangle ΔPMN is (ℓ, m)

$\ell = \frac{x_1 + \frac{1}{x_1} + x_2}{3}$; $m = \frac{y_1}{3}$

Normal at P: $\frac{x}{x_1} + \frac{y}{y_1} = 2$, put y = 0

$x = 2x_1 \Rightarrow x_2 = 2x_1$

$y_1^2 = x_1^2 - 1.$

$\Rightarrow \ell = \frac{3x_1 + \frac{1}{x_1}}{3}$

$$\frac{d\ell}{dx_1} = 1 - \frac{1}{3x_1^2} \Rightarrow \text{(A)}$$

$$m = \frac{y_1}{3}; m = \frac{\sqrt{x_1^2 - 1}}{3}$$

$$\frac{dm}{dy_1} = \frac{1}{3}; \frac{dm}{dy_1} = \frac{x_1}{3\sqrt{x_1^2 - 1}} \Rightarrow \text{(B) \& (D)}$$

55. The option(s) with the values of a and L that satisfy the following equation is (are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} = L \ ?$$

(A) a = 2, L = $\frac{e^{4\pi} - 1}{e^{\pi} - 1}$ (B) a = 2, L = $\frac{e^{4\pi} + 1}{e^{\pi} + 1}$ (C) a = 4, L = $\frac{e^{4\pi} - 1}{e^{\pi} - 1}$ (D) a = 4, L = $\frac{e^{4\pi} + 1}{e^{\pi} + 1}$

Ans. **AC**

Sol. $I_N = \int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt$

$$= \underbrace{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt}_{I} + \underbrace{\int_{\pi}^{2\pi} e^t(\sin^6 at + \cos^4 at)dt}_{II} + \underbrace{\int_{2\pi}^{3\pi} e^{at}(\sin^6 at + \cos^4 at)dt}_{III} + \underbrace{\int_{3\pi}^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}_{IV}$$

Put t = π + v

$$I_2 = \int_0^{\pi} e^{\pi} e^v(\sin^6 av + \cos^6 av)dv$$

$$= e^{\pi}\int_0^{\pi} e^v(\sin^6 av + \cos^4 av)dv$$

Put t = 2π + v

$$I_3 = e^{2p}\int_0^{\pi} e^v(\sin^6 av + \cos^4 av)dv$$

Put t = 3π + v

$$I_4 = e^{3p}\int_0^{\pi} e^v(\sin^6 av + \cos^4 av)dv$$

$I_N = I_1 + I_2 + I_3 + I_4$

$$= (1 + e^{\pi} + e^{2\pi} + e^{3\pi}) \int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt = \left(\frac{e^{4\pi} - 1}{e^{\pi} - 1}\right) I_D$$

$$\therefore \frac{I_N}{I_D} = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$$

a = 2, 4.

56. Let f, g: [–1, 2] → R be continuous functions which are twice differentiable on the interval (–1, 2). Let the values of f and g at the points –1, 0 and 2 be as given in the following table:

	x = –1	x = 0	x = 2
f(x)	3	6	0
g(x)	0	1	-1

In each of the intervals (–1, 0) and (0, 2) the function (f – 3g)" never vanishes. Then the correct statement(s) is(are)

(A) f '(x) – 3g'(x) = 0 has exactly three solutions in $(-1, 0) \cup (0, 2)$

(B) f '(x) – 3g'(x) = 0 has exactly one solution in (–1, 0)

(C) f '(x) – 3g'(x) = 0 has exactly one solution in (0, 2)

(D) f '(x) – 3g'(x) = 0 has exactly two solutions in (–1, 0) and exactly two solutions in (0, 2)

Ans. **BC**

Sol. Consider h(x) = f(x) – 3g(x) on [– 1, 0]

h(–1) = 3 = h(0) Rolle's Theorem

Applicable $\Rightarrow \exists\, c \in (-1, 0)$ such that h'(c) = 0

$\Rightarrow$ f '(c) – 3g'(c) = 0 for some $c \in (-1, 0)$

Clearly there exists only one $x = c \in (-1, 0)$ for which h'(x) = 0.

because if $\exists\, c_1 \neq c$ for which h'(x) = 0 then again by Rolle's Theorem

h"(x) = 0 for some $x \in (-1, 0)$

i.e. f "(x) – 3g"(x) = 0 for some $x \in (-1, 0)$ which is not possible $\Rightarrow$ (B)

Again h(0) = 3, h(2) = 3

again by same argument f '(x) – 3g'(x) = 0 for some x in (0, 2) and such x is unique. $\Rightarrow$ (C)

SECTION 3

(Maximum Marks : 16)

. This section contains TWO paragraphs

. Based on each paragraph, there will be TWO questions

. Each question has FOUR options (A), (B), (C) and (D). One or more than one of these four option(s) is(are) correct

. For each question darken the bubble(s) corresponding to all the correct option(s) is the ORS

. Marking scheme:

+ 4 If only the bubble (s) corresponding to all the correct option(s) is/are darkened

0 If none of the bubbles is darkened

– 2 In all other cases

PARAGRAPH - 1

Let n_1 and n_2 be the number of red and black balls, respectively, in box I. Let n_3 and n_4 be the number of red and black balls, respectively, in box II.

57. One of the two boxes, box I and box II, was selected at random and a ball was drawn randomly out of this box. the ball was found to be red. If the probability that this red ball was drawn from box II is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1, n_2, n_3 and n_4 is(are)

(A) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$ (B) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$

(C) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$ (D) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

Ans. **A, B**

58. A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer, is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1 and n_2 is(are)

(A) $n_1 = 4$ and $n_2 = 6$ (B) $n_1 = 2$ and $n_2 = 3$

(C*) $n_1 = 10$ and $n_2 = 20$ (D*) $n_1 = 3$ and $n_2 = 6$

Sol. I II

n_1R n_2B n_3R n_4B

(i) Probability $=\dfrac{\frac{1}{2}\times\frac{n_3}{n_3+n_4}}{\frac{1}{2}\times\frac{n_1}{n_1+n_2}+\frac{1}{2}\times\frac{n_3}{n_3+n_4}}=\dfrac{n_3(n_1+n_2)}{n_1(n_3+n_4)+n_3(n_1+n_2)}=\dfrac{1}{3}$

(A) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$

$\therefore$ L.H.S. $=\dfrac{5\cdot 6}{3\cdot 20+5\cdot 6}=\dfrac{30}{90}=\dfrac{1}{3}.$

Hence, (A) is correct

(B) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$

$\therefore$ L.H.S. $=\dfrac{10\times 9}{3\times 60+10\times 9}=\dfrac{90}{270}=\dfrac{1}{3}$

Hence, (B) is correct

(C) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$

$\therefore$ L.H.S. $=\dfrac{5\times 14}{8\times 25+5\times 14}=\dfrac{70}{270}\neq\dfrac{1}{3}$

(D) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

$\therefore$ L.H.S. $=\dfrac{5\times 18}{6\times 25+5\times 18}=\dfrac{90}{240}\neq\dfrac{1}{3}.$

$\therefore$ Answer is (A) & (B)

(ii) Probability $=\dfrac{n_1}{n_1+n_2}\times\dfrac{(n_1-1)}{(n_1+n_2-1)}+\dfrac{n_2}{n_1+n_2}\times\dfrac{n_1}{(n_1+n_2-1)}=\dfrac{1}{3}$

(A) $n_1 = 4, n_2 = 6$

$$\text{L.H.S.} = \frac{4}{10}\times\frac{3}{9}+\frac{6}{10}\times\frac{4}{9}=\frac{36}{90}\neq\frac{1}{3}$$

(B) $n_1 = 2, n_2 = 3$

$$\text{L.H.S.} = \frac{2}{5}\times\frac{1}{4}+\frac{3}{5}\times\frac{2}{4}=\frac{8}{20}\neq\frac{1}{3}$$

(C) $n_1 = 10, n_2 = 20$

$$\text{L.H.S.} = \frac{10}{30}\times\frac{9}{29}+\frac{20}{30}\times\frac{10}{29}=\frac{290}{870}=\frac{1}{3}$$

Hence, (C) is correct

(D) $n_1 = 3, n_2 = 6$

$$\text{L.H.S.} = \frac{3}{9}\times\frac{2}{8}+\frac{6}{9}\times\frac{3}{8}=\frac{24}{72}=\frac{1}{3}$$

Hence, (D) is correct.

$\therefore$ Answer is (C) & (D).

PARAGRAPH - 2

Let F: **R** → **R** be a thrice differentiable function. Suppose that F(1) = 0, F(3) = – 4 and F '(x) < 0 for all $x \in \left(\frac{1}{2}, 3\right)$. Let $f(x) \equiv x F(x)$ for all $x \in R$.

59. The correct statement(s) is(are)

(A) f ' (1) < 0

(B) f (2) < 0

(C) f '(x) $\neq$ 0 for any $x \in (1,3)$

(D) f ' (x) = 0 for some $x \in (1,3)$

Ans. **ABC**

60. If $\int_1^3 x^2F'(x)dx = -12$ and $\int_1^3 x^3F''(x)dx = 40$ then the correct expression(s) is(are)

(A) 9f '(3) + f '(1) – 32 = 0

(B) $\int_1^3 f(x)dx = 12$

(C) 9f '(3) – f '(1) + 32 = 0

(D) $\int_1^3 f(x)dx = -12$

Ans. **CD**

Sol.

(i) $\underbrace{F'(x) < 0}_{\text{decreasing function}}, \; x \in \left(\frac{1}{2}, 3\right)$

f(x) = x F(x)

F(1) = 0, F(3) = – 4

f '(x) = x F'(x) + F(x)

$f'(1) = F'(1) + F(1) = F'(1) < 0 \Rightarrow$ (A)

$f(2) = 2F(2)$

Since, F is decreasing and F (1) = 0 $\Rightarrow$ F (2) < 0

$\therefore$ f (2) = 2 F (2) < 0 $\Rightarrow$ **(B)**

Again $f'(x) = \underbrace{x F'(x)}_{<0} + \underbrace{F(x)}_{<0}$ $\forall x \in (1, 3)$

$\Rightarrow f'(x) < 0 \ \forall x \in (1, 3) \Rightarrow f'(x) \neq 0 \ \forall x \in (1, 3)$ $\therefore$ Ans. (C)

(ii) **[CD]**

$$\int_1^3 x^2F'(x)\,dx = -12; \int_1^3 x^3F''(x)\,dx = 40$$

$$\int_1^3 x^2F'(x)\,dx = x^2F(x)\Big|_1^3 - 2\int_1^3 xF(x)\,dx = -12$$

$$= -36 - 2\int_1^3 f(x)\,dx = -12 \Rightarrow \int_1^3 f(x)\,dx = -12 \Rightarrow (D)$$

Now, $\int_1^3 x^3F''(x)\,dx = x^3F'(x)\Big|_1^3 - 3\int_1^3 x^2F'(x)\,dx$

$$= 27\,F'(3) - F'(1) - 3\left[x^2F(x)\Big|_1^3 - 2\int_1^3 xF(x)\,dx\right]$$

$$= 27\,F'(3) - F'(1) - 3\left[9F(3) - F(1) - 2\int_1^3 f(x)\,dx\right]$$

$= 27F'(3) - 27F(3) - F'(1) + 3F(1) + 6 \times (-12)$

$= 27F'(3) + 108 - F'(1) - 72 = 9\,(3\,F'(3)) - F'(1) + 36$

$= 9\,(f'(3) - F(3)) - (f'(1) - F(1)) + 36$

$= 9\,f'(3) + 36 - f'(1) + 36 = 40$

$\Rightarrow 9f'(3) - f'(1) + 32 = 0 \Rightarrow$ (C)]